Intro to log geo.

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"A logarithmic structure is a "magic" by which a degenerate scheme begins to behave as being non-degenerate"
-Kato / Log. degenerations and Dieudonne theory

References:

- Kato / Log. str. of Fontaine-Illusie ← historical
- Abramovich et al. / Log. geo. & moduli } Intro.
- Temkin / Intro. to log. geo.
- Ogus / Lectures on log. alg. geo. ← full detail

(I) Motivation.

$$\overline{M}_{g,n} \quad - \text{DM (stack)}$$
$$\cup$$
$$M_{g,n} \quad \leftarrow \text{sm} \quad , \; D := \overline{M}_{g,n} \setminus M_{g,n} \quad \text{NCD}$$

↑ normal crossing divisor.

$$\text{NCD} \subset X \rightsquigarrow \text{log str. } X \rightsquigarrow \text{log str. } W = X \setminus D.$$
$$= M_{g,n}$$

$$\underset{\overline{M}_{g,n}}{\text{recover}} \quad \longleftarrow \quad M_{g,n} \text{ log str.}$$

Specifically,

$$\mathcal{L}M_{g,n} = \left\{ \begin{matrix} \text{log-sm} \\ \text{curves} \end{matrix} \right\} \leftarrow \text{cpt.}$$
$$\cup$$

$\{ \text{sm} \} = M_{g,n}$.

Moral: Easy comactification.

Convention: Rings / Monoids are commutative w/ unit.
$(k = \bar{k}, \ k = \mathbb{C})$

(II) Schemes.

Idea: Top.sp + local / infinitesimal data
$\underbrace{}_{\text{packaged in a sheaf of ring.}}$

Def. $\underset{\text{a ring}}{A} \rightsquigarrow \operatorname{Spec} A = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ ← str. sheaf.

$$\operatorname{Spec} A = \{ p \triangleleft A \mid \begin{smallmatrix} p \subsetneq A \\ \text{prime} \end{smallmatrix} \}$$

closed sets: $V(p) = \{ a \in \operatorname{Spec} A \mid p \subset a \}$

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \left\{ s : U \to \coprod_{p \in U} A_p \;\middle|\; \begin{array}{l} s(p) \in A_p \\ s \text{ locally quotient of a constant} \end{array} \right\}.$$

$\mathcal{O}_{\operatorname{Spec} A, p} = A_p$. ← the stalk

Def. Affine scheme $(X, \mathcal{O}_X) = X$ s.t.

top space $\operatorname{sp}(X)$; sheaf of rings

$X \simeq \operatorname{Spec} A$ for some A.

Def. Scheme $(X, \mathcal{O}_X)$ locally affine.

Def. Mor of schemes $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$

is a pair $f : sp(X) \longrightarrow sp(Y)$

$f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$.

Examples.

① $A = k$, $X = \operatorname{Spec} k$

$sp(X) = \{(0)\}$, $\mathcal{O}_X(X) = k$.

$$f : Y \longrightarrow X \quad \rightsquigarrow \quad \left\{ \begin{array}{l} f : sp(Y) \longrightarrow pt \\ f^\# : k \longrightarrow \mathcal{O}_Y(Y) \end{array} \right.$$

② $k = \mathbb{C}$, $A = k[t]$ $\rightsquigarrow$ $X = \operatorname{Spec}(\mathbb{C}[t]) = \mathbb{A}^1_{\mathbb{C}}$

$sp(X) = \{(0)\} \cup \{ m_a = (t-a) \mid a \in \mathbb{C} \}$

$\overline{\{(0)\}} = sp(X)$. $\quad \overline{\{m_a\}} = \{m_a\}$

③ $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$

$\hookrightarrow \mathbb{A}^2_{\mathbb{C}}$ $\quad sp(\mathbb{A}^2_{\mathbb{C}}) = \{(0)\} \cup \{ m_{a,b} = (x-a, y-b) \}$

closed

$\hookrightarrow \mathbb{A}^2_{\mathbb{C}}$ $\quad sp(\mathbb{A}^2_{\mathbb{C}}) = \{(0)\} \cup \{m_{a,b} = (x-a)(y-b)\}$

$\quad \| \qquad\qquad\qquad\qquad$ generic $\qquad\qquad$ closed

$\mathrm{Spec}\, \mathbb{C}[x,y]$ $\qquad\qquad \cup \{\text{irred. pol.} \}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad f$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$

nullstellensatz $\rightsquigarrow$ $\qquad \eta$

$\overline{\{\eta\}} = \{\eta\} \cup \{m_{a,b} \mid f(a,b)=0\}$

(4) $\mathbb{P}^n_k = \mathrm{Proj}\, k[x_0, \ldots, x_n]$.

<u>Rmk:</u> $\exists \quad Var \underset{ff}{\hookrightarrow} Sch$

Differentials.

<u>Def.</u> $Y \overset{i}{\hookrightarrow} X$ closed subscheme

$\rightsquigarrow$ the sheaf of ideals of Y is

$$\mathcal{I} := \ker\left(i^\#: \mathcal{O}_X \to i_* \mathcal{O}_Y\right)$$

<u>Def.</u> Let $f: X \to Y$, $\quad \Delta: X \to X \times_Y X$

$\rightsquigarrow \Delta(X)$ closed subscheme of $W \underset{op.}{\subset} X \times_Y X$

$$\mathcal{I} := \mathcal{I}_{\Delta(X)},$$

The sheaf of rel. differentials is

$$\Omega_{X/Y} = \Delta^* \mathcal{I}/\mathcal{I}^2.$$

<u>Example:</u> $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, $\underline{f: A \to B}$

$f: X \to Y$

$$\rightsquigarrow X \times_Y X \simeq \operatorname{Spec}(B \otimes_A B)$$

$$\Rightarrow \Delta(X) = \ker\left(\begin{array}{ccc} B \otimes_A B & \longrightarrow & B \\ b \otimes b' & \longmapsto & bb' \end{array} \right) =: I$$

$\mathcal{I}/\mathcal{I}^2$ = the sheaf assoc. w/ I/I^2

As derivation:

$$d : B \longrightarrow I/I^2$$

$$db = 1 \otimes b - b \otimes 1 \qquad (\text{mod } I^2)$$

$$\rightsquigarrow \quad I/I^2 \xrightarrow{\sim} \langle db \rangle_{b \in B} \Big/ \begin{array}{l} d(b+b') = db + db' \\ d(bb') = bdb' + b'db \\ d|_A = 0 \end{array} =: \Omega_{B/A}$$

"Kähler differentials."

Special case: $B = A[x_1, \dots, x_n]$

$$\rightsquigarrow \Omega_{B/A} = \langle dx_1, \dots, dx_n \rangle_B$$

III Log. schemes.

Idea: log scheme = scheme + ambient data

= top + infi. data + "ghost" extrinsic

$= \underset{\text{sp}}{\text{top}} + \underbrace{\text{inf'l data}}_{\text{"intrinsic data"}} + \text{"ghost" extrinsic data.}$

$M_{g,n} \subset \overline{M}_{g,n}$, $D = \overline{M}_{g,n} \setminus M_{g,n}$

sheaf of rings

sheaf of monoids

Def. pre-log scheme $X = (\underline{X}, \mathcal{M}_X, \alpha = \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$

underlying sheaf $(\underline{X}, \mathcal{O}_{\underline{X}})$

a sheaf of monoids on $\underline{X}$

mor' of monoids.

log scheme = pre-log X s.t. $\underbrace{\alpha^{-1}(\mathcal{O}^*_{\underline{X}})}_{\mathcal{M}^*_X} \simeq \mathcal{O}^*_{\underline{X}}$.

The char. monoid of scheme $\overline{\mathcal{M}}_X := \mathcal{M}_X / \mathcal{O}^*_{\underline{X}}$

Notation: May write

$$\alpha(q) = x \longleftrightarrow \alpha^q = x$$

Remark: Given pre-log $\alpha : \mathcal{M} \to \mathcal{O}_X$, can "logify" by push-out

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}^*_{\underline{X}}) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \end{array}$$

$$\begin{array}{ccc} \downarrow & \perp & \downarrow \\ \mathcal{O}_{\underline{X}}^* & \longrightarrow & \mathcal{M}^a \end{array}.$$

Examples.

① trivial log str:

$$\underline{X} \text{ sch.} \rightsquigarrow \mathcal{M}_{triv} := \mathcal{O}_{\underline{X}}^*$$

$$\alpha = \text{inclusion}$$

Rmk: a) $\mathcal{M}_{triv}$ is the initial obj of $\underbrace{\begin{pmatrix} \text{log str.} \\ \text{on } \underline{X} \end{pmatrix}}_{(*)}$

b) $\exists$ $Sch \overset{}{\underset{ff}{\hookrightarrow}} \mathcal{L}Sch$

② full log str.: $\mathcal{M}_{term} = \mathcal{O}_{\underline{X}}$,

$\alpha = id$

- terminal obj. in $(*)$.

③ log pt.

$\underline{X} = \operatorname{Spec} k$

$\mathcal{O}_{\underline{X}} = k.$

$\boxed{\mathbb{N} = \mathbb{Z}_{\geq 0}}$

$$\mathcal{M}_X : k^* \oplus \mathbb{N} \longrightarrow \mathcal{O}_X = k$$

$$(c, n) \longmapsto \begin{cases} c & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$= c \cdot 0^n$$

$$\left(0^n = \begin{cases} 1 & n = 0 \\ 0 & o/w \end{cases} \right)$$

Check:

$$\alpha^{-1}(\mathcal{O}_{\underline{X}}^*) = \alpha^{-1}(k^*)$$

$$\alpha^{-1}(\mathcal{O}^{\alpha}_{\underline{X}}) = \alpha^{-1}(k^*)$$
$$= k^* \oplus \{0\} \simeq k^* = \mathcal{O}^*_{\underline{X}}.$$

$\left(0^n = \begin{cases} 1 & n=0 \\ 0 & o/w \end{cases} \right)$

④ Divisors:

$\underline{X} = \operatorname{Spec} k[x_1, \dots, x_n]$, $D = V(x_1 \cdot \dots \cdot x_r)$

$r \leq n$.

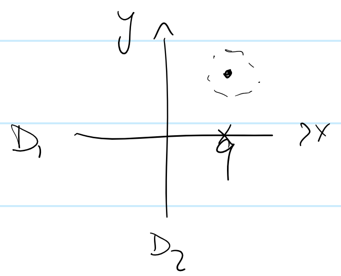
$W := \underline{X} \setminus D \xhookrightarrow{i} \underline{X}$ (NCD)

$$\mathcal{M}_X := \mathcal{O}_{\underline{X}} \cap i_* \mathcal{O}^*_W$$

i.e., $\mathcal{M}_X(U) = \{ g \in \mathcal{O}_{\underline{X}}(U) \mid g|_{U \setminus D} \in \mathcal{O}^*_{\underline{X}}(U \setminus D) \}$

↳ sub-example:

$\underline{X} = \operatorname{Spec} \mathbb{C}[x,y]$. , $D = \{xy = 0\}$.



$$\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}^*_{\underline{X}}$$

$$\overline{\mathcal{M}}_{X,q} = \begin{cases} 0 & q \notin D \\ \mathbb{N} & q \in D_1 \setminus \{0\},\ q \in D_2 \setminus \{0\} \\ \mathbb{N} \oplus \mathbb{N} & q = 0 \end{cases}$$

⑤ Degenerations

$$\begin{array}{c} X \\ \downarrow \\ \mathbb{A}^1_k \\ t \end{array}$$

flat

X_0 nc

X_t sm , $t \neq 0$

$$\begin{array}{c} xy - t \\ \downarrow \\ t \end{array}$$

$D = X_0 \rightsquigarrow$ log str on X

$\rightsquigarrow$ log str. on X_0.

Differentials $\quad f: X \to Y$

$$\Omega^1_{X/Y} = \Omega_{X/Y} = \Omega_{\underline{X}/\underline{Y}} \oplus \mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp} \big/ \sim$$

Example: $X = \operatorname{Spec} R$, $\quad R = k[x_1, \dots, x_n] \big/ (x_1 \cdot \cancel{\;} \cdot x_r) \leftarrow$

$$M_X = k^* \oplus \mathbb{N}^r \longrightarrow R$$
$$e_j \longmapsto x_j$$

$$Y = \log \text{pt} = \operatorname{spec} k$$

$$M_Y = k^* \oplus \mathbb{N} \longrightarrow k$$
$$(c, n) \longmapsto c \cdot 0^n$$

$$\rightsquigarrow \Omega_{X/Y} = \left\langle \frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_n \right\rangle_{\mathcal{O}_X} \Big/ \frac{dx_1}{x_1} + \dots + \frac{dx_r}{x_r} = 0$$

$$\boxed{\frac{dx_1}{x_1} = d(\log x_1)}$$

$$\text{"} \Omega_X(\log D) = \langle d(\log f) \mid \underline{f \in M} \rangle \text{"}$$

$X = \operatorname{Spec} A$

P monoid

$$f: X \to Y \quad \rightsquigarrow \quad \underbrace{f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X}_{\text{logify}}$$

$$f^* \mathcal{M}_Y$$
